

TANGENT BUNDLE OF \mathbb{P}^2 AND MORPHISM FROM \mathbb{P}^2 TO $\text{Gr}(2, \mathbb{C}^4)$

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ABSTRACT. In this note we study the image of \mathbb{P}^2 in $\text{Gr}(2, \mathbb{C}^4)$ given by tangent bundle of \mathbb{P}^2 . We show that there is component \mathcal{H} of the Hilbert scheme of surfaces in $\text{Gr}(2, \mathbb{C}^4)$ with no point of it corresponds to a smooth surface.

Keywords: Projective plane; Tangent bundle; Morphisms; Grassmannian.

1. INTRODUCTION

Let \mathbb{P}^2 denote the projective plane over the field of complex numbers \mathbb{C} and $\text{Gr}(2, \mathbb{C}^4)$ Grassman variety of two dimensional quotients of the vector space \mathbb{C}^4 .

The aim of this paper is to study the image of \mathbb{P}^2 by non constant morphisms $\mathbb{P}^2 \rightarrow \text{Gr}(2, \mathbb{C}^4)$ obtained by tangent bundle $T_{\mathbb{P}^2}$ of \mathbb{P}^2 . The bundle $T_{\mathbb{P}^2}$ is generated by sections and hence it is generated by four(= $\text{rank}(T_{\mathbb{P}^2}) + \dim(\mathbb{P}^2)$) independent global sections. Any set S of four independent generating sections of $T_{\mathbb{P}^2}$ defines a morphism

$$\phi_S : \mathbb{P}^2 \rightarrow \text{Gr}(2, \mathbb{C}^4),$$

such that the $\phi_S^*(Q) = T_{\mathbb{P}^2}$, where Q is the universal rank two quotient bundle on $\text{Gr}(2, \mathbb{C}^4)$:

$$\mathcal{O}_{\text{Gr}(2, \mathbb{C}^4)}^4 \rightarrow Q \rightarrow 0.$$

According to a result of Tango [4], if $\phi : \mathbb{P}^2 \rightarrow \text{Gr}(2, \mathbb{C}^4)$ is an imbedding then the pair of Chern classes $(c_1(\phi^*(Q)), c_2(\phi^*(Q)))$ is equal to one of the following pairs: $(H, 0)$, (H, H^2) , $(2H, H^2)$ or $(2H, 3H^2)$, where H is the ample generator of $H^2(\mathbb{P}^2, \mathbb{Z})$.

Since $c_1(T_{\mathbb{P}^2}) = 3H$ and $c_2(T_{\mathbb{P}^2}) = 3H^2$, the morphism ϕ_S defined by a set S of four independent generating sections of $T_{\mathbb{P}^2}$ is not an imbedding. Thus, it is natural to ask, does there exists a set S of four independent generating sections of $T_{\mathbb{P}^2}$ for which ϕ_S is generically injective? In this direction we have the following (Theorem 3.6):

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Theorem 1.1. *For general choice of an ordered set S of four independent generating sections of $T_{\mathbb{P}^2}$ the morphism*

$$\phi_S : \mathbb{P}^2 \rightarrow \text{Gr}(2, \mathbb{C}^4)$$

is generically injective.

We also, show that in fact one can find an ordered set of generators S of $T_{\mathbb{P}^2}$ the morphism is an immersion i.e., the morphism induces an injection on all the tangent spaces.

As by product of our result we obtain the following (Theorem 4.2):

Theorem 1.2. *There is an irreducible component \mathcal{H} of the Hilbert scheme of surfaces in $\text{Gr}(2, \mathbb{C}^4)$ no point which corresponds to a smooth surface.*

2. THE TANGENT BUNDLE OF \mathbb{P}^2 .

The tangent bundle of \mathbb{P}^2 fits in an exact sequence called the “Euler sequence”:

$$(1) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{O}_{\mathbb{P}^2}(1)^3 \rightarrow T_{\mathbb{P}^2} \rightarrow 0.$$

This exact sequence together with the fact $H^1(\mathcal{O}_{\mathbb{P}^n}) = 0$, implies that $\dim H^0(T_{\mathbb{P}^2}) = 8$, where H^i denotes the i th sheaf cohomology group. Since the rank two bundle $T_{\mathbb{P}^2}$ on \mathbb{P}^2 is ample and generated by sections, a minimal generating set of independent sections has cardinality four. Any set S of four independent generators of $T_{\mathbb{P}^2}$ gives to an exact sequence:

$$0 \rightarrow E_S \rightarrow \mathcal{O}_{\mathbb{P}^2}^4 \rightarrow T_{\mathbb{P}^2} \rightarrow 0.$$

This in turn corresponds to a morphism $\phi_S : \mathbb{P}^2 \rightarrow \text{Gr}(2, \mathbb{C}^4)$, where $\phi_S(x) = \{\mathbb{C}^2 = \mathcal{O}_{\mathbb{P}^2}^4|_x \rightarrow T_{\mathbb{P}^2}|_x \rightarrow 0\}$.

3. THE MAIN RESULT

Note that in the “Euler sequence” (1) the injective map

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{O}_{\mathbb{P}^2}(1)^3$$

is given by the section $v = (X, Y, Z)$, where X, Y, Z is the standard basis of $H^0(\mathcal{O}_{\mathbb{P}^2}(1))$. For a section v_i of $\mathcal{O}_{\mathbb{P}^2}(1)^3$ we denote w_i the image section of $T_{\mathbb{P}^2}$ under the surjection

$$\mathcal{O}_{\mathbb{P}^2}(1)^3 \rightarrow T_{\mathbb{P}^2} \rightarrow 0$$

in (1). Let $\tilde{S} = (v_1, v_2, v_3, v_4)$ be an ordered set of four linearly independent sections of $\mathcal{O}_{\mathbb{P}^2}(1)^3$ and $S = (w_1, w_2, w_3, w_4)$ be the corresponding

ordered set of sections of $T_{\mathbb{P}^2}$. Clearly, the set S is generating set of independent sections of $T_{\mathbb{P}^2}$ if and only if $\tilde{S} \cup \{v\}$ is a generating set of independent sections of $\mathcal{O}_{\mathbb{P}^2}(1)^3$.

Lemma 3.1. *Let $v_1 = (X, 0, 0), v_2 = (0, Y, 0), v_3 = (Y, Z, X), v_4 = (Z, X, Y)$ be four sections of $\mathcal{O}_{\mathbb{P}^2}(1)^3$ and $\tilde{S} = (v_i | 1 \leq i \leq 4)$. Then the ordered set \tilde{S} with $\{v = (X, Y, Z)\}$ is a generating set of independent sections of $\mathcal{O}_{\mathbb{P}^2}(1)^3$. Hence the corresponding ordered set of sections $S = (w_1, w_2, w_3, w_4)$ generate $T_{\mathbb{P}^2}$, where w_i is the image of v_i under the map given in exact sequence (1).*

Proof: Clearly, v generate a subspace of $\mathcal{O}_{\mathbb{P}^2}(1)^3$ dimension one at every point of \mathbb{P}^2 . Hence w_i, w_j is not independent at a point $p \in \mathbb{P}^2$ if and only if the section $v_{ij} = v_i \wedge v_j \wedge v$ of $\mathcal{O}_{\mathbb{P}^2}(3)$ vanishes at p . Thus, if the six independent sections $\{v_i \wedge v_j \wedge v | 1 \leq i < j \leq 4\}$ has no common zero implies \tilde{S} with $\{v = (X, Y, Z)\}$ is a generating set of independent sections of $\mathcal{O}_{\mathbb{P}^2}(1)^3$. Note that $v_{12} = XYZ, v_{13} = X(Z^2 - XY), v_{14} = X(XZ - Y^2), v_{23} = Y(X^2 - YZ), v_{24} = Y(YX - Z^2), v_{34} = 3XYZ - (X^3 + Y^3 + Z^3)$. It is easy to see that the set $\{v_{12}, v_{13}, v_{14}, v_{23}, v_{24}, v_{34}\}$ of sections of $\mathcal{O}_{\mathbb{P}^2}(3)$ has no common zero in \mathbb{P}^2 and hence the ordered set \tilde{S} with $\{v = (X, Y, Z)\}$ is a generating set of independent sections of $\mathcal{O}_{\mathbb{P}^2}(1)^3$. Hence the corresponding ordered set of sections $S = (w_1, w_2, w_3, w_4)$ generate $T_{\mathbb{P}^2}$. \square

Lemma 3.2. *Let $f : \mathbb{P}^2 \rightarrow \mathbb{P}^n$ be a non constant morphism and $f^*(\mathcal{O}_{\mathbb{P}^n}(1)) = \mathcal{O}_{\mathbb{P}^2}(m)$. Assume that there exists a linear subspace W of codimension two such that $W \cap f(\mathbb{P}^2)$ consists of exactly m^2 points. Then the morphism f is generically injective.*

Proof: Note as the morphism f is non constant $f^*(\mathcal{O}_{\mathbb{P}^n}(1)) = \mathcal{O}_{\mathbb{P}^2}(m)$ with $m > 0$ and hence is ample. This means f is finite map. Set $r = \deg(f)$, the number of elements $f^{-1}(f(x))$ for a general $x \in \mathbb{P}^2$. If d to be the degree of $f(\mathbb{P}^2)$ in \mathbb{P}^n then it is easy to see that $m^2 = d.r$. On the other hand the assumption, $W \cap f(\mathbb{P}^2)$ consists of exactly m^2 points, implies $d \geq m^2$. Thus we must have $d = m^2$ and $r = \deg(f) = 1$. Thus f is generically injective. \square

Lemma 3.3. *With the notations of Lemma(3.1), the surjection of vector bundles on \mathbb{P}^2*

$$\mathcal{O}_{\mathbb{P}^2}^4 \rightarrow T_{\mathbb{P}^2}$$

given by S defines a generically injective morphism

$$\phi_S : \mathbb{P}^2 \rightarrow \text{Gr}(2, \mathbb{C}^4).$$

Proof: Let $p : \text{Gr}(2, \mathbb{C}^4) \rightarrow \mathbb{P}^5$ be the Plucker imbedding given by the determinant of the universal quotient bundle. Then $p \circ \phi_S$ is given by

$$\begin{aligned} (x; y; z) &\mapsto \\ &(xyz; x(z^2 - xy); x(xz - y^2)); y(x^2 - yz); y(xy - z^2); \\ &3xyz - (x^3 + y^3 + z^3)). \end{aligned}$$

To prove the map ϕ_S is generically injective it is enough to prove the map $p \circ \phi_S$ is so. Set (Z_0, \dots, Z_5) as the homogeneous coordinates of \mathbb{P}^5 and W be the codimension two subspace of \mathbb{P}^5 defined by $Z_0 = 0 = Z_5$. Then $W \cap p \circ \phi_S(\mathbb{P}^2)$ is equal to

$$\{(0, -\omega^i, 1, 0, 0, 0); (0, 0, 0, -\omega^i, 1, 0); (0, \omega^i, 1, \omega^i, 1, 0) | 1 \leq i \leq 3\},$$

where ω is a primitive cube root of unity. Note that

$$(p \circ \phi_S)^*(\mathcal{O}_{\mathbb{P}^5}(1)) = \mathcal{O}_{\mathbb{P}^2}(3).$$

Hence, the required result follows from Lemma(3.2). \square

Remark 3.4. We show (see Lemma 4.1) that $p \circ \phi_S$ is an immersion. i.e., the induced linear map on the tangent space at every point of \mathbb{P}^2 is injective and one to one except finitely many points.

Next we recall the following [See, Lemma(3.13)[1]]:

Lemma 3.5. Let X and Y be two irreducible projective varieties. Let T be an irreducible quasi-projective variety and $t_0 \in T$ be a point. Let

$$F : X \times T \rightarrow Y$$

be a morphism. Assume that $F_t := F|_{X \times t} : X \rightarrow Y$ is finite for all $t \in T$ and F_{t_0} is a birational onto its image. Then there is an open subvariety U of T such that $t_0 \in U$ and for $t \in U$ the morphism F_t is birational onto its image.

Proof: For the sake of completeness we reproduce the proof here. Consider the morphism $G = F \times \text{Id}_T : X \times T \rightarrow Y \times T$. Then the assumption F_t is finite implies the morphism G is finite and proper. Hence $\mathcal{G} = G_*(\mathcal{O}_{X \times T})$ is coherent sheaf of $\mathcal{O}_{Y \times T}$ modules. Let $Z \subset Y \times T$ be the subvariety on which the sheaf $G_*(\mathcal{O}_{X \times S})$ is supported. Then clearly the map $p : Z \rightarrow T$, restriction of the natural projection, is surjective. The section $1 \in \mathcal{O}_{X \times T}$ gives an inclusion of \mathcal{O}_Z in \mathcal{G} . Let $\mathcal{F} = \mathcal{G}/\mathcal{O}_Z$. Let $Z_1 \subset Y \times T$ be the subvariety on which the sheaf \mathcal{F} supported. Let $q : Z_1 \rightarrow T$ be the natural projection and let $U = \{t \in T | \dim q^{-1}(t) < \dim(X)\}$ then we see that by semi continuity [See, page 95, Exercise (3.22) [3]], U is an open subset and is non-empty as $t_0 \in U$. For $t \in U$ the morphism F_t is an isomorphism on $X \times t - G^{-1}(q^{-1}(t))$. Since G is finite $G^{-1}(q^{-1}(t))$ is proper closed subset

of $X \times t$ and hence the morphism F_t is birational onto its image. This proves the Lemma. \square

Theorem 3.6. *For a generic choice of an ordered set S of four independent generating sections of $T_{\mathbb{P}^2}$ the morphism*

$$\phi_S : \mathbb{P}^2 \rightarrow \text{Gr}(2, \mathbb{C}^4)$$

is generically injective.

Proof: It is easy to see that the ordered set of four sections S generating $T_{\mathbb{P}^2}$ is an irreducible quasi projective variety. In fact it is an open subvariety of the affine space V^4 , where $V = H^0(T_{\mathbb{P}^2})$. The theorem at once follows from Lemma(3.5), if we show the existence of one S for which ϕ_S is generically injective. But the existence of one such S follows from Lemma(3.3). \square

4. AN EXAMPLE

The result of the previous section can be used give an example of a component of a Hilbert Scheme of $Gr(2, \mathbb{C}^4)$ with out any point corresponding to a smooth surface.

Lemma 4.1. *The morphism $p \circ \phi_S : \mathbb{P}^2 \rightarrow \mathbb{P}^5$ of Lemma(3.2) is an immersion i.e., the induced linear map on the tangent space at every point of \mathbb{P}^2 is injective. Moreover, $p \circ \phi_S$ one to one except*

$$S_1 = \{(1; 0; 0), (0; 1; 0), (0; 0; 1)\} \mapsto (0; 0; 0; 0; 0; 1)$$

and

$$S_2 = \{(1; 1; 1), (\omega; \omega^2; 1), (\omega^2; \omega; 1)\} \mapsto (1; 0; 0; 0; 0; 0),$$

where ω is a primitive cube root of unity.

Proof: Let X, Y, Z be the homogeneous coordinates functions on \mathbb{P}^2 and $Z_0, Z_1, Z_2, Z_3, Z_4, Z_5$ be the homogeneous coordinates functions on \mathbb{P}^5 . Clearly under the morphism $p \circ \phi_S : \mathbb{P}^2 \rightarrow \mathbb{P}^5$ the set S_1 maps to $(0; 0; 0; 0; 0; 1)$ and the set S_2 maps to $(1; 0; 0; 0; 0; 0)$. Note that the lines $X = 0, Y = 0$, and $Z = 0$ mapped to nodal cubics $Z_0 = Z_1 = Z_2 = Z_3^3 + Z_4^3 - Z_3 Z_4 Z_5 = 0$, $Z_0 = Z_3 = Z_4 = Z_1^3 + Z_2^3 + Z_1 Z_2 Z_5 = 0$, and $Z_0 = Z_1 - Z_4 = Z_2 + Z_3 = Z_1^3 + Z_2^3 - Z_1 Z_2 Z_5 = 0$ respectively. Thus we can conclude that the morphism $p \circ \phi_S$ is an immersion on these three lines. On the complement of these lines the morphism $p \circ \phi_S$ can be described as

$$(x, y) \mapsto (1/y - x, x/y - y, x - y/x, y - 1/x, 3 - x^2/y - y^2/x - 1/xy)$$

from $\mathbb{C}^2 - \{xy = 0\} \rightarrow \mathbb{C}^4$. If (x, y) and (x_1, y_1) maps to the same point then we get the following equations:

$$(2) \quad 1/y - x = 1/y_1 - x_1$$

$$(3) \quad x/y - y = x_1/y_1 - y_1$$

$$(4) \quad y/x - x = y_1/x_1 - x_1$$

$$(5) \quad 1/x - y = 1/x_1 - y_1.$$

The equations 2 and 5 gives us

$$\frac{xy - 1}{y} = \frac{x_1y_1 - 1}{y_1}; \quad \frac{xy - 1}{x} = \frac{x_1y_1 - 1}{x_1}.$$

Since $xy \neq 0$ and $x_1y_1 \neq 0$ we see that either $xy - 1 \neq 0$ and $(x, y) = (x_1, y_1)$ or $xy - 1 = 0 = x_1y_1 - 1$. Hence we get $(p \circ \phi_S)$ is one to one outside the set $\{(1, 1), (\omega, \omega^2), (\omega^2, \omega)\}$ and this is mapped to $(0, 0, 0, 0, 0)$. The assertion about the immersion of the given morphism

$$\mathbb{C}^2 - \{xy = 0\} \rightarrow \mathbb{C}^4$$

can be checked by looking at the two by two minors of the below jacobian matrix of the morphism:

$$\begin{pmatrix} -1 & 1/y & 1 + y/x^2 & 1/x^2 & -2x/y + y^2/x^2 + 1/x^2y \\ -1/y^2 & -x/y^2 - 1 & -1/x & 1 & x^2/y^2 - 2y/x + 1/xy^2 \end{pmatrix}$$

□

Theorem 4.2. *Let \mathcal{H} be the irreducible component of the Hilbert scheme of $\text{Gr}(2, \mathbb{C}^4)$ containing the point corresponding to the image surface of the morphism*

$$\phi_S : \mathbb{P}^2 \rightarrow \text{Gr}(2, \mathbb{C}^4)$$

of 3.3. Then no point of \mathcal{H} corresponds to a smooth surface.

Proof: Let $p : \text{Gr}(2, \mathbb{C}^4) \rightarrow \mathbb{P}^5$ be the Plucker imbedding. Since $(p \circ \phi_S)^*(\mathcal{O}_{\mathbb{P}^5}(1)) = \mathcal{O}_{\mathbb{P}^2}(3)$ and by Lemma 4.1 the morphism $p \circ \phi_S$ is an imbedding outside finite set of points. Moreover, general hyperplane section of $(p \circ \phi_S)(\mathbb{P}^2)$ in \mathbb{P}^5 is smooth curve of genus one. If a point of the irreducible \mathcal{H} corresponds to a smooth surface Y then it has to have the same cohomology class as that of $(p \circ \phi_S)(\mathbb{P}^2)$ namely $(3, 6) \in H^4(\text{Gr}(2, \mathbb{C}^4), \mathbb{Z})$. Also, the general hyperplane section of $p(Y)$ has to be a smooth curve of genus one. But according to the classification of smooth surfaces of type $(3, 6)$ in $\text{Gr}(2, \mathbb{C}^4)$ (see, [2, Theorem 4.2]) implies that there are no smooth surface of type $(3, 6)$ with hyperplane section a smooth curve of genus one. This contradiction proves that no point of \mathcal{H} corresponds to a smooth surface. □

Remark 4.3. *The component \mathcal{H} of the Hilbert Scheme in Theorem 4.2 is reduced irreducible of dimension 23. In fact computing the normal sheaf associated to the morphism ϕ_S of Lemma 3.3 and counting the dimension of space of all such morphisms we see that \mathcal{H} is a reduced irreducible of dimension 23.*

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